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AFOSR-75-2793

AFOSR-TR-78-0020

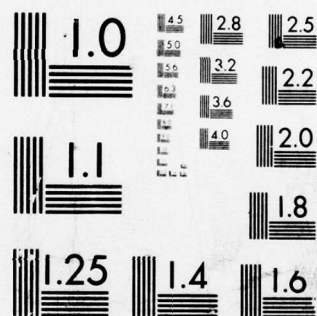
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On Feedback Stabilization of Time Varying Discrete Linear Systems\*

by

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Abstract

Results are given for stabilizing time varying discrete linear systems by means of a feedback control stemming from a receding horizon concept and a minimum quadratic cost with a fixed terminal constraint. The results parallel those recently obtained for continuous time systems [8] and extend a well known method of Kleinman for stabilizing discrete fixed linear systems [7].

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\*This research was supported in part by the Air Force Office of Scientific Research under Grant AFOSR-75-2793C.

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# I. Introduction

Consider a completely uniformly controllable and observable linear time varying discrete system

$$x_{i+1} = \phi_i x_i + B_i u_i \quad (1)$$

$$y_i = C_i x_i \quad (2)$$

where  $x_i \in R^n$  is the state vector,  $(\phi_i, B_i, C_i)$  are given  $n \times n$ ,  $n \times m$  and  $p \times n$  time varying matrices, and the state transition matrix  $\phi_i$  is assumed to be non-singular for all  $i$ . There are very few methods for constructing a stabilizing feedback control,  $u_i = L_i x_i$ , in the time varying case, the most familiar of which is the steady state optimal control for the quadratic cost

$$J = \lim_{i_f \rightarrow \infty} \sum_{i=i_0}^{i_f} y_i' Q_i y_i + u_i' R_i u_i \quad (3)$$

where  $Q_i \geq 0$  and  $R_i > 0$  [1-3]. Control laws based on a receding horizon notion in minimizing control energy subject to a moving terminal constraint have been developed by Thomas [4,5] in the case of time invariant systems. These controls add new insight into a method for stabilizing a fixed linear system due to Kleinman [6,7]. Recently, a generalization of the control law based on a receding horizon notion has been reported in [8]. In addition to the extension to time varying systems, this generalization includes the possibility of weighting the state vector through a nonnegative weighting matrix  $Q(t)$  which is analogous to the  $Q_i$  matrix in (3). The purpose of this note is to indicate the parallel nature of these results for time varying discrete systems and to point out the advantage in computing the gain matrix for this control law in relation to the steady state control for (3).

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## II. Main Result

We consider a feedback control law for the time varying system (1) - (2) which can be interpreted as optimal for the "receding horizon" quadratic cost

$$J = \sum_{k=i}^{i+N-1} y_k' Q_k y_k + u_k' R_k u_k \quad (4)$$

subject to the moving terminal constraint

$$x_{i+N} = 0. \quad (5)$$

When a non-null weighting matrix  $Q_k$  is chosen in (4), we assume that it is represented by  $Q_k = D_k' D_k$  for appropriately chosen matrix  $D_k$ . The optimal control for (4) - (5) can be determined using standard minimization techniques with a Lagrange multiplier to handle (5). Assuming the fixed horizon length  $N$  in (4) is chosen sufficiently long, the feedback form of this control can be expressed in the following way

$$u_i = -R_i^{-1} B_i' P^{-1}(i+1, i+1+N) \phi_i x_i \quad (6)$$

where the inverse of the double indexed  $P^{-1}$  matrix is defined implicitly through the solution to the discrete time Riccati type equation

$$\begin{aligned} P(k, j) = & \phi_k^{-1} P(k+1, j) \phi_k'^{-1} - \phi_k^{-1} P(k+1, j) \phi_k'^{-1} C_k' D_k' [I \\ & + D_k C_k \phi_k^{-1} P(k+1, j) \phi_k'^{-1} C_k' D_k']^{-1} D_k C_k \phi_k^{-1} P(k+1, j) \phi_k'^{-1} \\ & + B_{k-1} R_{k-1}^{-1} B_{k-1}', \quad P(j, j) = 0. \end{aligned} \quad (7)$$

The  $n \times n$  symmetric matrix  $P(i+1, i+1+N)$  is obtained by summing (7) backward from  $k = i+1+N$  to  $k = i+1$  for any given  $j = i+1+N$ . Although the control law (6) - (7) is obtained by summing a Riccati equation over a finite time interval (in contrast with the infinite time summation interval required



for the minimum solution of the steady state quadratic cost (3)), it is asserted here that this control renders the feedback system (1) asymptotically stable under appropriate controllability and observability conditions. The following theorem is the discrete time analog of Theorem 2.1 in [8]. The integers  $\ell_c$  and  $\ell_o$  in the statement of this theorem pertain to the uniform bounds on the controllability and observability Gramians in that complete uniform controllability of the pair  $(\phi_i, B_i)$  implies the existence of a fixed integer  $\ell_c$  such that

$$\alpha_1 I \leq \sum_{k=i}^{i+\ell_c-1} \phi(i, k+1) B_k B_k' \phi'(i, k+1) \leq \alpha_2 I$$

$$\phi(i, j) \triangleq \prod_{k=j}^{i-1} \phi_k$$

for all  $i$  and for some positive scalars  $(\alpha_1, \alpha_2)$ . The integer  $\ell_o$  is similarly defined using the complete uniform observability of the pair  $(\phi_i, C_i)$ .

**Theorem** (1) Assume that  $R_i$  and  $Q_i$  satisfy  $\alpha_3 I \leq R_i \leq \alpha_4 I$  and  $0 \leq Q_i \leq \alpha_6 I$  for positive scalars  $(\alpha_3, \alpha_4, \alpha_6)$  and for all  $i$ . Under the conditions that the pair  $(\phi_i, B_i)$  is uniformly completely controllable and  $C_i$  is bounded such that  $\|C_i\| \leq \alpha_7$  for all  $i$ , the system (1) - (2) with feedback control (6) is uniformly asymptotically stable when the horizon length  $N$  is chosen to satisfy  $\ell_c + 1 \leq N < \infty$ . (Note:  $Q_i$  and  $C_i$  can be identically zero.)

(2) Suppose  $R_i$  and  $Q_i$  satisfy  $\alpha_3 I \leq R_i \leq \alpha_4 I$  and  $\alpha_5 I \leq Q_i \leq \alpha_6 I$  for positive scalars  $(\alpha_3, \alpha_4, \alpha_5, \alpha_6)$  and for all  $i$ . Under the conditions that the pairs  $(\phi_i, B_i)$  and  $(\phi_i, C_i)$  are completely uniformly controllable and observable, the system (1) - (2) with feedback control (6) is uniformly asymptotically stable when the horizon length  $N$  is chosen to satisfy:  $1 + \max(\ell_c, \ell_o) \leq N \leq \infty$ .

Proof Outline: The proof parallels that given in [8] for continuous time systems in that consideration is given to the adjoint system of (1) with control (6), viz.

$$\tilde{x}_{i+1} = [\phi_i - B_i R_i^{-1} B_i' P^{-1}(i+1, i+1+N) \phi_i]^{-1} \tilde{x}_i \quad (8)$$

$$\Delta F_i' \tilde{x}_i$$

together with the associated scalar valued function

$$V(\tilde{x}, i) = \tilde{x}' \phi_i^{-1} P(i+1, i+1+N) \phi_i^{-1} \tilde{x}. \quad (9)$$

Similar to the approach in [8], the feedback system (1) and (6) will be asymptotically stable if and only if the adjoint system (8) is asymptotically unstable. The main difference between the continuous time proof in [8] and the proof of the above Theorem is the judicious choice in Lyapunov function (9) for the discrete time case. Using a detailed analysis which can be found in the complete report [9], it can be shown that  $V(\tilde{x}, i)$  in (9) is uniformly bounded above and below by positive scalars  $(\alpha_8, \alpha_9)$  such that  $\alpha_8 \|\tilde{x}\| \leq V(\tilde{x}, i) \leq \alpha_9 \|\tilde{x}\|$  for all  $i$ . This involves establishing uniform lower and upper bounds on the matrix  $P(i, i+N)$ . Next, it can be shown that the difference  $\{V(\tilde{x}_{i+1}, i+1) - V(\tilde{x}_i, i)\}$  is nondecreasing along solutions of (8) and, more importantly, is uniformly bounded below in the sense that there exists a positive scalar  $\alpha_{10}$  so that

$$V(\tilde{x}_{i+1}, i+1) - V(\tilde{x}_{i_0}, i_0) \geq \alpha_{10} \|\tilde{x}_{i_0}\|$$

for all  $i \geq i_0 + N$ , under the hypothesis of the theorem. This inequality hinges on establishing the property of the matrix solution to (7) that  $P(k, j_1) \leq P(k, j_2)$  for  $k \leq j_1 \leq j_2$ , and the property  $\tilde{P}^{-1}(k, j_1) \geq \tilde{P}^{-1}(k, j_2)$  when

$k < j_1 \leq j_2$  for a related matrix  $\tilde{P}^{-1}(k,j) \triangleq K(k,j)$  which is defined implicitly by the solution to

$$K(k,j) = \phi_k' K(k+1,j) \phi_k - \phi_k' K(k+1,j) B_k [R_k + B_k' K(k+1,j) B_k]^{-1} B_k' K(k+1,j) \phi_k + C_k' Q_k C_k$$

with the boundary condition  $K(j,j) = 0$ . As pointed out in [9], the matrix  $K(k,j)$  is the same Ricatti gain matrix which appears in the solution to the standard discrete regulator problem (without an end point constraint). This fact leads to a relation between the control (6) and the standard regulator control in that it is possible to establish the following inequalities (details presented in [9]):

$$x_o' K(i_o, i_o + N) x_o \leq \sum_{k=i_o}^{i_o + N - 1} y_k' Q_k y_k + u_k' R_k u_k \leq x_o' \tilde{P}^{-1}(i_o, i_o + N) x_o$$

where  $u_k$  is presumed specified by (6). This shows that the control (6) tends to the same control as the steady state quadratic cost (3) when the horizon length  $N \rightarrow \infty$ , since (6) is an asymptotically stable control and  $\tilde{P}^{-1}(k,j) = K(k,j)$ . Thus in this sense the control (6) can be interpreted as an approximation to the standard linear quadratic cost problem (3) which has the property that its underlying Riccati equation is summed over a finite time interval while still guaranteeing asymptotic stability.

### III. Concluding Remarks

1. In the case of a time invariant system and constant weighting matrices, i.e.  $(\phi, B; C, Q=D'D, R)$  all constant, the control (6) reduces to

$$u_i = -R^{-1} B' P^{-1}(N) \phi x_i \quad (10)$$



where  $P(N)$  can be obtained from

$$\begin{aligned} P(k+1) = & \phi^{-1}P(k)\phi'^{-1} - \phi^{-1}P(k)\phi'^{-1}C'D'[I \\ & + DC\phi^{-1}P(k)\phi'^{-1}C'D']^{-1}DC\phi^{-1}P(k)\phi'^{-1} \\ & + BR^{-1}B', \quad P(0) = 0. \end{aligned} \quad (11)$$

This control includes the results of [5] and [7] as special cases by choosing  $Q=0$ , or  $C=0$ ; viz, (11) can be summed in closed form as

$$P(N) = \sum_{k=0}^{N-1} \phi^{-k}BR^{-1}B'\phi'^{-k} \quad (12)$$

which coincides with the gain matrices of [5] and [7]. In [10] it is shown that the matrix  $\phi$  in (12) can be singular for a controllable single input system, while still facilitating a stabilizing feedback control law.

2. Other results which parallel the continuous time version in [8] are:

(i) The degree of relative stabilization of the control (6) can be guaranteed in the sense that  $\|x_i\| \rightarrow 0$  as least as fast as  $(\frac{1}{\alpha})^i$  as  $i \rightarrow \infty$  for a chosen  $\alpha > 1$ , provided the matrix  $\phi_k$  in (7) is replaced by  $\alpha\phi_k$  for all  $k$ . (ii) The dual problem in filtering theory for the fixed terminal minimum energy problem, (4) - (5), is the standard linear filter which yields minimum error covariance at time  $i$  with a completely unknown moving initial condition at time  $(i-N)$ . These results together with the rigorous proof of the Main Result are contained in [9].

3. As in the continuous time version [8], the proof for the main result has been rather indirect involving the adjoint system (8) and a suitable Lyapunov function. However, a direct proof in the time invariant case can be obtained (as in the time invariant case in [8]) using the Lyapunov function

$V(x) = x' \Phi^{-1} P(N) \Phi^{-1} x$  for the system  $x_{i+1} = [\Phi - BR^{-1} B' P^{-1}(N) \Phi] x_i$ . Perhaps a more direct proof exists for the time varying case as well, though we have not found one.

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7. AUTHOR(s) <b>W. H. /Kwon A. E. /Pearson</b>		6. PERFORMING ORG. REPORT NUMBER
9. PERFORMING ORGANIZATION NAME AND ADDRESS Brown University Division of Engineering Providence, RI 02912		8. CONTRACT OR GRANT NUMBER(s) <b>15 AFOSR-75-2793</b>
11. CONTROLLING OFFICE NAME AND ADDRESS Air Force Office of Scientific Research/NM Bolling AFB DC 20332		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS 61102F <b>17</b> <b>16 2304(A)</b>
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)		12. REPORT DATE <b>14 Nov 77</b>
		13. NUMBER OF PAGES 8 <b>12 TOP.</b>
		15. SECURITY CLASS. (of this report) UNCLASSIFIED
16. DISTRIBUTION STATEMENT (of this Report)  Approved for public release; distribution unlimited.		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE
17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)		
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